The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Final Exam ***Solutions 3.0***

Date: December 19, 2018 Course: EE 313 Evans

Name: **World,** **Girl Meets**
Last, First

- The exam is scheduled to last three hours.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please turn off all cell phones.
- No headphones are allowed.
- All work should be performed on the exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation.

Problem	**Point Value**	**Average score**	**Standard Deviation**	**Topic**	**Solution**
1	14	11.8	1.9	Heart and Soul of Continuous Time	*Smackle*
2	12	10.4	3.0	Discrete-Time System Identification	*Farkle*
3	12	7.4	2.6	Continuous-Time Convolution	*Topanga*
4	12	8.9	2.1	Discrete-Time Feedback System	*Auggie*
5	12	7.5	2.7	Continuous-Time Circuit Analysis	*Lucas*
6	12	6.3	3.6	Bluetooth Receiver	*Maya*
7	12	6.6	3.0	Continuous-Time Equalization	*Cory*
8	14	6.2	3.9	Discrete-Time Audio Effects	*Riley*
Total	100	65.0	14.8		

Note: 71 students took the final exam. The median score was 67. A 12-point offset was added to each student's score to bring the median score to 79 and the average score to 77.0.

Problem 1. Heart and Soul for Continuous-Time Signals and Linear Systems. *14 points*.

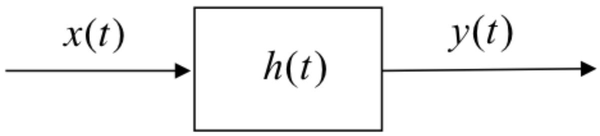
(a) **LTI Systems**. Consider a continuous-time linear time-invariant system with input signal x(t), impulse response $h(t)$ and output signal $y(t)$. 9 *points*.

i. Give the relationship for $y(t)$ to $x(t)$ and $h(t)$ involving only operations in the time domain.

ii. Give the relationship for $Y(j\omega)$ to $X(j\omega)$ and $H(j\omega)$ using only operations in the Fourier (frequency) domain.

iii. Give the relationship for $Y(s)$ to $X(s)$ and $H(s)$ using only operations in the Laplace domain.

time domain



$y(t) =$

Fourier domain $X(j\omega)$ $H(j\omega)$ $Y(j\omega)$ $Y(j\omega) =$

Laplace domain $X(s)$ $H(s)$ $Y(s)$ $Y(s) =$

***y*(*t*) = *h*(*t*) * *x*(*t*)** **-OR-** ***y*(*t*) = *x*(*t*) * *h*(*t*)**

***Y*(*jω*) = *H*(*jω*) *X*(*jω*)** **-OR-** ***Y*(*jω*) = *X*(*jω*) *H*(*jω*)**

***Y*(*s*) = *H*(*s*) *X*(*s*)** **-OR-** ***Y*(*s*) = *X*(*s*) *H*(*s*)**

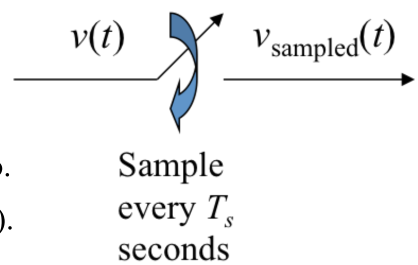
SPFirst Ch. 9 p. 327-328

SPFirst Ch. 11 p. 257-258

SPFirst Ch. 16 p. 20

$\mathbf{ROC}\{\boldsymbol{Y(s)}\} = \mathbf{ROC}\{\boldsymbol{X(s)}\} \cap \mathbf{ROC}\{\boldsymbol{H(s)}\}$

(b) **Sampling & Aliasing.** *5 points.*



i. *Sampling*. Consider sampling modeled as an instantaneous closing and opening of a switch every T_s seconds.

When the sampling switch is open, assume $v_{\text{sampled}}(t)$ is zero.

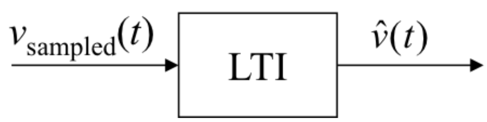
Give a time-domain expression for $v_{\text{sampled}}(t)$ in terms of $v(t)$.

$$v_{sampled}(t) = v(t) \sum_{n=-\infty}^{\infty} \delta(t - n\,T_s)$$

SPFirst Sec. 12-3.1

Lecture Slide 16-3

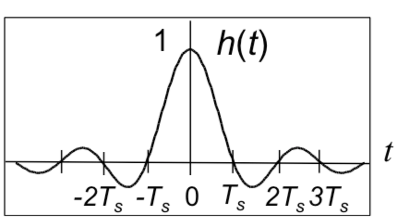
ii. *Reconstruction*. Describe the LTI system needed to extract an estimate of $v(t)$ from $v_{\text{sampled}}(t)$.

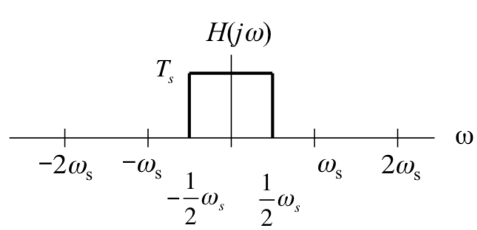


The estimate of $v(t)$ is denoted as $\hat{v}(t)$.

Over what frequencies is the estimate of $v(t)$ accurate?

The LTI system would extract the spectrum of *V*(*jω*) from $V_{\text{sampled}}(j\omega)$. The LTI systems would be ideal lowpass filter with impulse response *h*(*t*) = sin(π*t*/T_s) / (π*t*/T_s). The Sampling Theorem says to choose $f_s > 2\,f_{\max}$ and we can divide by 2 on both sides to obtain $f_{\max} < (1/2)\,f_s$. The estimate of *v*(*t*) is accurate over $-(1/2)\,f_s < f < (1/2)\,f_s$.





SPFirst Sec. 12-3.3

Lecture Slide 16-5

Problem 2. Discrete-Time System Identification. *12 points*. **Using Deconvolution**

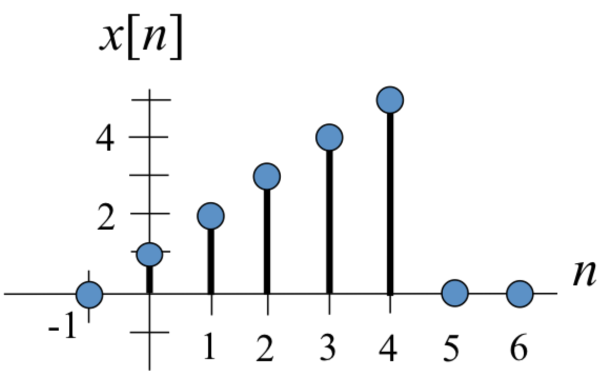
You are given a causal discrete-time linear time-invariant (LTI) system with unknown impulse response $h[n]$ to analyze.

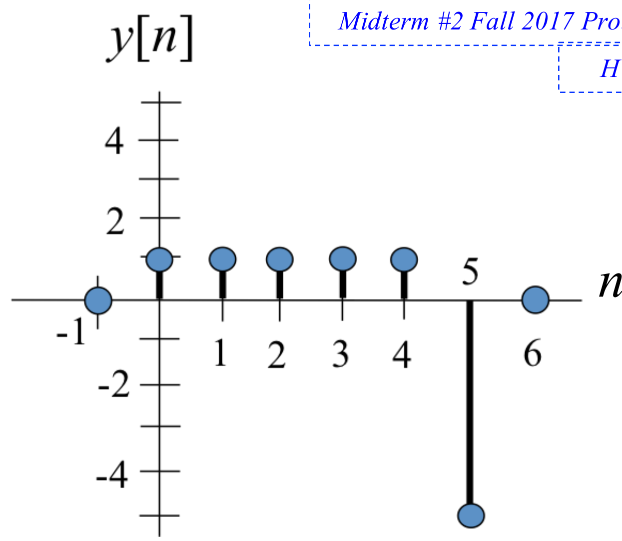
When the five-sample causal signal $x[n]$ given below is input into the unknown system, the response $y[n]$ is six samples long and causal, as shown below.

Midterm #2 Fall 2018 Problem 2.2

Midterm #2 Fall 2017 Problem 2.1

HW 6.3(b)





x = [1 2 3 4 5 0 0];

y = [1 1 1 1 -5 0];

(a) Find $h[n]$. *9 points.*

Answer #1: **Because $x[n]$ and $y[n]$ are finite in length, the length of $y[n]$ is the length of $h[n]$ plus the length $x[n]$ minus one due to discrete-time convolution. Hence, $h[n]$ has two coefficients.**

Since $x[n]$ and $y[n]$ are causal, $h[n]$ is also causal. That is, the convolution of two causal signals would always give a causal result. So, $h[n] = h[0]\,\delta[n] + h[1]\,\delta[n-1]$.

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] = h[0]\, x[n] + h[1]\, x[n-1]$$

Since the system is LTI, initial conditions are zero. We start by computing one sample at a time:

$1 = y[0] = h[0]\, x[0] = h[0]$ and hence $h[0] = 1$.

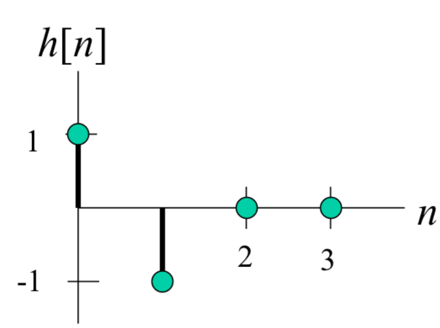
$1 = y[1] = h[0]\, x[1] + h[1]\, x[0] = (1)(2) + h[1]\,(1)$ and hence $h[1] = -1$.

We can check this in MATLAB using `h = [1 -1]; x = [1 2 3 4 5 0 0]; conv(h, x);`

Answer #2: **We can compute the transfer function in the z-domain and then apply the inverse z-transform to obtain the impulse response. Since $Y(z) = H(z)\, X(z)$,**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} - 5\, z^{-5}}{1 + 2\, z^{-1} + 3\, z^{-2} + 4\, z^{-3} + 5\, z^{-4}} = 1 - z^{-1}$$

using polynomial long division. The MATLAB command `deconv(y, x)` **will compute this. After applying the inverse z-transform to $H(z)$, we obtain $h[0] = 1$ and $h[1] = -1$.**



(b) Plot $h[n]$. *3 points*. **See plot on the right.**

Problem 3. Continuous-Time Convolution. *12 points*.

Convolve the two-sided continuous-time signals

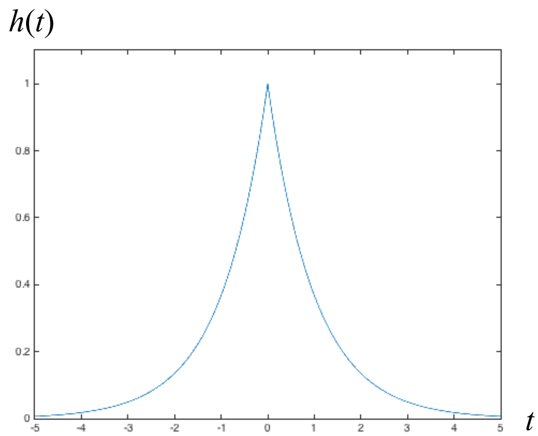
$x(t) = \cos(\omega_0 t)$ and $h(t) = e^{-a\,|t|}$

where a is real-valued and $a > 0$.

Both signals are defined for $-\infty < t < \infty$.

A plot of $h(t)$ over a finite interval of time is shown on the right for $a = 1$.

Please solve this problem for a general positive real value for a.



As per Problem #1, we can work any continuous-time problem in time, frequency (Fourier), or generalized frequency (Laplace) domains.

<u>Time-Domain Approach:</u> We apply the convolution definition directly

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda) d\lambda = \int_{-\infty}^{\infty} e^{-a\,|\lambda|} \cos\big(\omega_0 (t-\lambda)\big)\, d\lambda$$

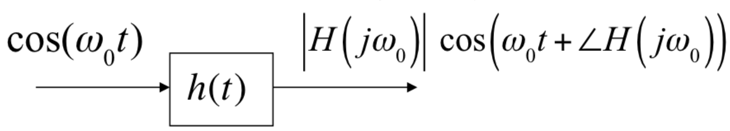
We can write $e^{-a\,|t|} = e^{a\,t}\, u(-t) + e^{-a\,t}\, u(t)$ **provided that** $u(0) = \frac{1}{2}$**. Now we have**

$$y(t) = \int_{-\infty}^{0} e^{a\,\lambda} \cos\big(\omega_0 (t-\lambda)\big)\, d\lambda + \int_{0}^{\infty} e^{-a\,\lambda} \cos\big(\omega_0 (t-\lambda)\big)\, d\lambda$$

Okay, let's see if there is an easier alternate method.

<u>Laplace Domain Approach:</u> A two-sided cosine does not have a Laplace transform…

<u>Fourier Domain Approach:</u> We can view this problem as $x(t)$ being an input signal into a linear time-invariant (LTI) system with impulse response given by $h(t)$. From Lecture Slide 14-6:



The continuous-time Fourier transform of $h(t)$ is available on page 326 of *SPFirst* :

$$H(j\omega) = \frac{2a}{a^2+\omega^2} \text{ where } |H(j\omega)| = \frac{2a}{a^2+\omega^2} \text{ and } \angle H(j\omega) = 0$$

because $a > 0$. Note that $H(j\omega)$ is a lowpass filter. Now that we know $H(j\omega)$,

$$y(t) = \frac{2a}{a^2+\omega_0^2} \cos(\omega_0 t)$$

Or one could also find the Fourier transform of $h(t)$ by using the definition after writing

$$h(t) = e^{-a\,|t|} = e^{a\,t}\, u(-t) + e^{-a\,t}\, u(t) \text{ to obtain } H(j\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

A student used the symbolic toolbox in MATLAB to compute the Fourier transform:

```
syms a t w
h = exp(-a*abs(t));
fourier(h, t, w)
```

Problem 4. Discrete-Time Feedback System. *12 points.*

Consider a discrete-time linear time-invariant (LTI) system with input signal $x[n]$ and output signal $y[n]$ that is governed by the following second-order difference equation for $n \geq 0$:

$$y[n] = 1.8\, y[n-1] - K\, y[n-2] + x[n]$$

where K is a real-valued constant.

(a) What are the initial conditions of the system and what values should they have? *3 points.*

From page 198 of SPFirst, a necessary condition for a system to have LTI properties is that it must be "at rest". That is, the initial conditions of the system must be zero.

We can find the initial conditions of the system by computing the first output values:

$$\boldsymbol{y[0] = 1.8\, y[-1] - K\, y[-2] + x[0]}$$

$x[0]$ and $y[0]$ are initial values of input signal $x[n]$ and output signal $y[n]$, respectively. Neither is an initial condition of the system. Hence $y[-1] = 0$ and $y[-2] = 0$.

(b) Derive the transfer function $H(z)$ for the system, which will depend on K. *3 points.*

Take the z-transform of both sides of the equation:

$$Y(z) = 1.8\, z^{-1} Y(z) - K\, z^{-2}\, Y(z) + X(z)$$

Next, collect $Y(z)$ terms on left-hand side of the equation:

$$Y(z) - 1.8\, z^{-1}\, Y(z) + K\, z^{-2}\, Y(z) = X(z)$$

$$\left(1 - 1.8\, z^{-1} + K\, z^{-2}\right) Y(z) = X(z)$$

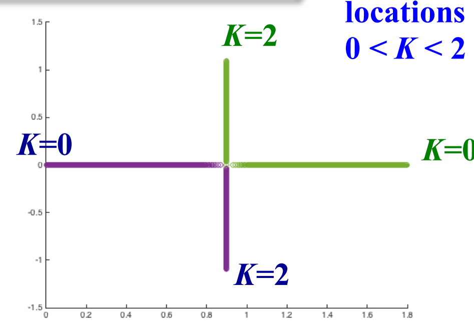
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 1.8\, z^{-1} + K\, z^{-2}} = \frac{z^2}{z^2 - 1.8\, z + K}$$

$H(z)$ has two zeros at the origin ($z = 0$) and poles at

$$\frac{-1.8 \pm \sqrt{(-1.8)^2 - 4K}}{2} = \frac{-1.8 \pm \sqrt{(-2 \times 0.9)^2 - 4K}}{2} = 0.9 \pm \sqrt{0.81 - K}$$

Region of convergence is complex z plane outside a circle of the larger pole radius: $|z| > \max\{\, |p_0|, |p_1|\, \}$

Plot of pole locations for $0 < K < 2$




```
K = 0 : 0.001 : 2;
root1 = 0.9 - sqrt(0.81 - K);
root2 = 0.9 + sqrt(0.81 - K);
hold on;
scatter(real(root1), imag(root1));
scatter(real(root2), imag(root2));
hold off;
```

(c) Give the range of values for K for which the system is bounded-input bounded-output (BIBO) stable. *3 points.*

Both poles must be inside unit circle for BIBO stability. For $K \leq 0.81$, poles are real-valued:

$$-1 < 0.9 - \sqrt{0.81 - K} \text{ and } 0.9 + \sqrt{0.81 - K} < 1$$

The left inequality gives $K > -2.8$ and the right one gives $K > 0.8$. Hence, $K > 0.8$.

For $K > 0.81$, the poles are complex-valued: $0.9 + j\sqrt{K - 0.81}$.

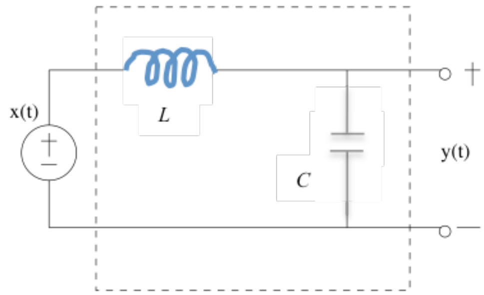
$$\left|0.9 + j\sqrt{K - 0.81}\right| < 1 \text{ which means } \sqrt{(0.9)^2 + (K - 0.81)} < 1$$

By squaring both sides, $(0.9)^2 + (K - 0.81) < 1$ which means $K < 1$. So, $0.8 < K < 1$.

(d) Describe the possible frequency selectivity (lowpass, highpass, bandpass, bandstop, allpass or notch) that the system could exhibit for different values of K for which the system is BIBO stable. *3 points.* **For $0.8 < K < 0.81$, poles are real-valued between 0.8 and 1.0, not inclusive. Lowpass. At $K = 0.81$, there is double pole at $z = 0.9$, which is also means a lowpass response. As K increases from 0.81 to 1, the pole separation increases and response becomes bandpass.**

Problem 5. Continuous-Time Circuit Analysis. *12 points*. **Continuous-Time Oscillator.**

Consider the following analog continuous-time circuit with input voltage $x(t)$ and output voltage $y(t)$:



SPFirst Ch. 16 page 31

The initial voltage across the capacitor is 0 V and the initial current in the inductor is 0 A; hence, the circuit is a linear time-invariant system.

(a) Using the voltage drop around the loop

$$x(t) - L\frac{d}{dt}i(t) - \frac{1}{C}\int_{0^-}^{t} i(t)dt = 0$$

take the Laplace transform of both sides of the equation to find the relationship between $X(s)$ and $I(s)$. $I(s)$ is the Laplace transform of the current $i(t)$. *3 points*.

$\boldsymbol{X(s) - Ls\, I(s) - \frac{1}{Cs} I(s) = 0}$ **which becomes** $\boldsymbol{X(s) = Ls\, I(s) + \frac{1}{Cs} I(s) = \left(Ls + \frac{1}{Cs}\right) I(s)}$

Because of the (1/*s*) term due to integration in time, the region of convergence is $\boldsymbol{Re\{s\} > 0}$**.**

(b) Using the formula for the voltage across the capacitor

$$y(t) = \frac{1}{C}\int_{0^-}^{t} i(t)dt$$

take the Laplace transform of both sides and substitute the expression for $I(s)$ obtained in part (a) to obtain the transfer function $H(s)$ in the Laplace domain so that $H(s) = Y(s) / X(s)$. *3 points*.

$\boldsymbol{Y(s) = \frac{1}{Cs}\, I(s) = \frac{1}{Cs}\frac{X(s)}{\left(Ls + \frac{1}{Cs}\right)} = \frac{1}{LCs^2+1} X(s)}$ **which leads to** $\boldsymbol{H(s) = \frac{Y(s)}{X(s)} = \frac{1}{LCs^2+1} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{LC}}}$

(c) What is the impulse response $h(t)$? *3 points.*

From the second-to-last row in the Laplace transform table in SPFirst Chapter 16 Laplace Transforms on page 16 with *a* = 0, the impulse response is a sinusoidal signal:

$\boldsymbol{h(t) = \omega_0 \sin(\omega_0\, t)\, u(t)}$ **where** $\boldsymbol{\omega_0 = \frac{1}{\sqrt{LC}}}$.

(d) Is the system bounded-input bounded-output stable? *3 points*.

<u>Answer #1:</u> From part (a), the region of convergence for the transfer function is Re{*s*} > 0, which does not include the imaginary axis. The system is not BIBO stable.

<u>Answer #2:</u> For an LTI system to be BIBO stable, the poles must be in left-hand side of the *s* plane. The pole locations are at $\boldsymbol{s = j\frac{1}{\sqrt{LC}}}$ **and** $\boldsymbol{s = -j\frac{1}{\sqrt{LC}}}$. **The system is not BIBO stable.**

Problem 6. Bluetooth Receiver. *12 points.* **Software Receiver Design.**

SPFirst p. 360-361

Bluetooth operates in the 2400-2499 MHz unlicensed frequency band.

At any given time, Bluetooth will transmit on one of 79 channels, and each channel is 1 MHz wide.

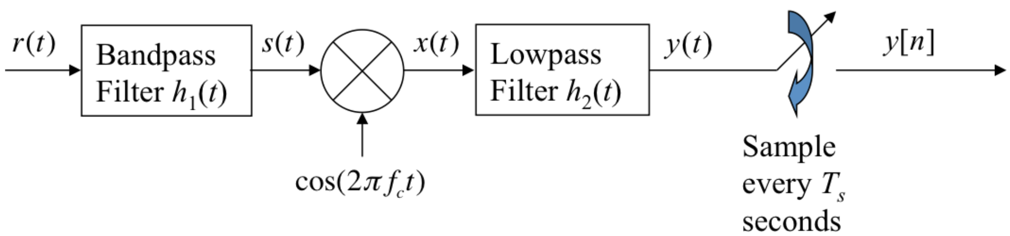
Channel k begins at $(2402 + k)$ MHz where $k = 0, 1, \ldots, 78$.

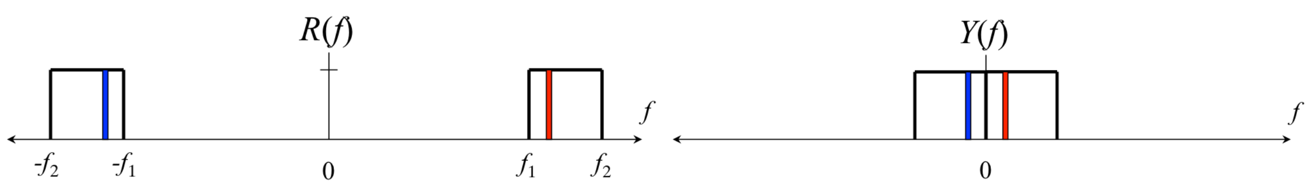
Bluetooth changes the 1 MHz channel on which it operates 1600 times/second to avoid interference.

A Bluetooth receiver has two subsystems in cascade. The first subsystem involves continuous-time signal processing blocks and the second subsystem involves discrete-time signal processing blocks.

(a) **The continuous-time signal processing** blocks are given below, where $r(t)$ is the received radio frequency signal. In the plot for $R(f)$, one of the 1 MHz channels is shaded, and its counterpart in negative frequencies is also shaded. Demodulation produces $y(t)$, whose spectrum $Y(f)$ is below.

Let $f_1 = 2400$ MHz and $f_2 = 2499$ MHz. What is the demodulating frequency f_c? *3 points*.





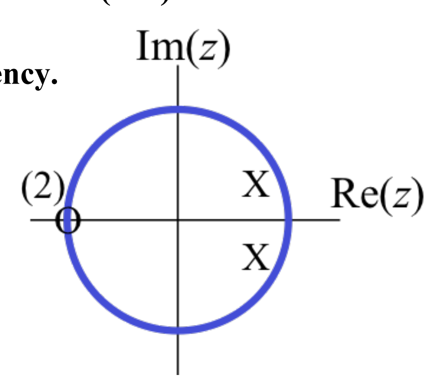
$f_c = f_1$. Analog/RF front end performs sinusoidal amplitude demodulation to shift the positive frequency band in $R(f)$ to the left by f_1 and negative frequency band in $R(f)$ to the right by f_1.

We divide demodulation for channel k into two stages. The first stage is an analog/RF front end with fixed functionality that passes all Bluetooth channels to the second stage. The second stage selects channel k and can be programmed in software. In laptops and smart phones, a single chip implements Wi-Fi and Bluetooth with a shared analog/RF front end.

(b) The first **discrete-time signal processing block** is filtering. Design a **second-order** linear time-invariant (LTI) infinite impulse response (IIR) filter to extract channel k from $y[n]$. Use a sampling rate of $f_s = 200$ MHz. Give formulas for, and plot, the two poles and two zeros. *9 points*.

Due to the analog/RF front end in part (a), channel k in $y(t)$ resides between $(k+2)$ MHz and $(k+3)$ MHz. Center frequency is at $(k+2.5)$ MHz in continuous-time frequency and $\omega_k = 2\pi \frac{(k+2.5)\text{ MHz}}{200\text{ MHz}} = 2\pi \frac{k+2.5}{200}$ in discrete-time frequency.

For the second-order discrete-time IIR filter, the poles are at $p_0 = 0.9\, e^{j\omega_k}$ and $p_1 = 0.9\, e^{-j\omega_k}$. For small values of k, poles would be close to 0 rad/sample. To avoid strong interaction between zeros and poles, place the two zeros at $z = -1$.



Problem 7. Continuous-Time Equalization. *12 points.*

Lecture Slide 18-5	*SPFirst p. 277*

When sound waves propagate through air, or when electromagnetic waves propagate through air, the waves are absorbed, reflected and scattered by objects in the environment.

In the transmission of sound waves over the air in a room from an audio speaker to a microphone, we will model the direct path from the speaker to the microphone as having zero delay, and a one-bounce path from the speaker to an object and then to the microphone having delay t_1.

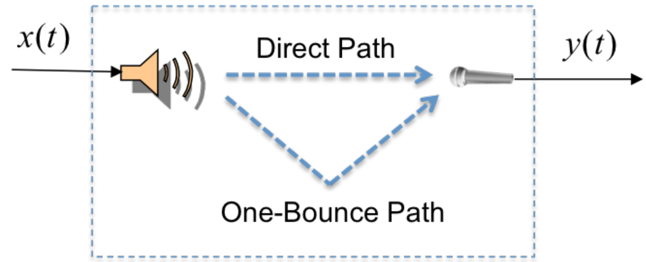
This single reflection is a type of echo.

We model the signal $y(t)$ at the output of the microphone as

$$y(t) = x(t) - \alpha\, x(t - t_1)$$

where α is a real-valued constant and $t_1 > 0$.

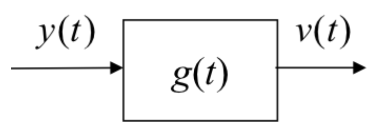
We model that system that connects $x(t)$ and $y(t)$ as linear and time-invariant (LTI).



(a) Derive a formula for the impulse response $h(t)$. *3 points.* $\boldsymbol{h(t) = \delta(t) - \alpha\, \delta(t - t_1)}$.

(b) Find transfer function in the Laplace domain $H(s)$. *3 points.* $\boldsymbol{H(s) = 1 - \alpha\, e^{-s\,t_1}}$ **for all** $\boldsymbol{s}$.

(c) We add an LTI filter at the microphone output to remove as much of the echo as possible. Design the continuous-time filter by giving its transfer function $G(s)$ in the Laplace domain. The filter must be bounded-input bounded-output (BIBO) stable. *6 points.*



a. Case I. $\alpha < 0$. **We would like to have $H(s)\, G(s) = 1$:**

$$G(s) = \frac{1}{H(s)} = \frac{1}{1 - \alpha\, e^{-s\,t_1}}$$

When $\alpha < 0$, the denominator cannot be zero.

To see this, we substitute $s = \sigma + j\omega$ into the denominator to obtain

$$1 - \alpha\, e^{-(\sigma + j\omega)\,t_1} = 1 - \alpha\, e^{-\sigma\,t_1}\, e^{-j\,\omega\,t_1} = 1 - \alpha\, e^{-\sigma\,t_1}\left(\cos(\omega\, t_1) - j \sin(\omega\, t_1)\right)$$

In order for the denominator to be zero, both real and imaginary components must be zero at the same time. The real component is $1 - \alpha\, e^{-\sigma\,t_1} \cos(\omega\, t_1)$ and imaginary component is $\alpha\, e^{-\sigma\,t_1} \sin(\omega\, t_1)$. The $e^{-\sigma\,t_1}$ term is in the interval [0, 1] because $\sigma \geq 0$ for BIBO stability and $t_1 > 0$. The real component is always positive because $\alpha < 0$.

b. Case II. $\alpha = 0$. **The one-bounced path is zeroed out. $H(s) = 1$. Hence, $G(s) = 1$.**

c. Case III: $\alpha > 0$. **(Based on a student's solution.)**

When $G(s) = \frac{1}{1 - \alpha\, e^{-s\,t_1}}$ as in Case I, the denominator goes to zero when

$$1 - \alpha\, e^{-s\,t_1} = 0 \quad \Rightarrow \quad \alpha\, e^{-s\,t_1} = 1 \quad \Rightarrow \quad s = -\frac{1}{t_1} \ln \frac{1}{\alpha} = \frac{\ln \alpha}{t_1}$$

When $0 < \alpha < 1$, pole is negative and real, so $G(s)$ is BIBO stable: $G(s) = \frac{1}{1 - \alpha\, e^{-s\,t_1}}$

When $\alpha = 1$, pole is on the imaginary axis and not BIBO stable: $G(s) = \frac{1}{1 - 0.99\, e^{-s\,t_1}}$

When $\alpha > 1$, we invert α to get negative real pole and BIBO stability: $G(s) = \frac{1}{1 - \frac{1}{\alpha} e^{-s\,t_1}}$

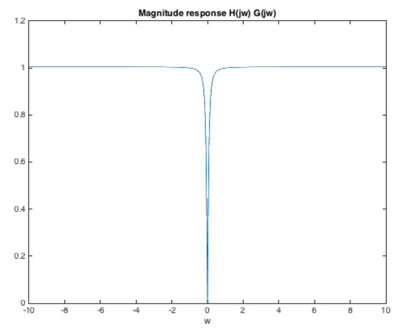
Here is more analysis for Case III from problem 7(c), and this analysis is beyond the analysis expected for problem 7 during the final exam.

When $\alpha = 1$, the pole is on the imaginary axis, which is not BIBO stable. For the equalizer, we'll use the same form of $1 / H(s)$ but use a value in (0,1) in place of $\alpha = 1$, e.g. $G(s) = \frac{1}{1-0.99\, e^{-s\, t_1}}$

This will cause the cascade of $H(s)$ and $G(s)$ to be in a notch filter configuration:

$$H(s)\, G(s) = \frac{1 - e^{-s\, t_1}}{1 - 0.99\, e^{-s\, t_1}}$$

In the cascaded transfer function $H(s)\, G(s)$, the zero will be at $s = 0$ and the pole will be a negative real number close to zero: $s = \frac{\ln \alpha}{t_1} = -\frac{0.0101}{t_1}$ and $t_1 > 0$. Here's the resulting magnitude response after substituting $s = j\,\omega$ into $H(s)\, G(s)$:




```
w = -10 : 0.01 : 10;
t1 = 0.1;
HG = (1 - exp(-j*w*t1)) ./ (1 - 0.99*exp(-j*w*t1));
plot(w, abs(HG));
xlabel('w [rad/s]');
title('Magnitude response H(jw) G(jw)');
```

The equalizer G(s) gives a cascade $H(s)\, G(s)$ that is allpass except for frequencies around 0 rad/s.

When $\alpha > 1$, we invert α to get negative real pole and BIBO stability: $G(s) = \frac{1}{1-\frac{1}{\alpha}e^{-s\, t_1}}$

The cascade of $H(s)$ and $G(s)$ will give an all-pass filter (i.e. a constant magnitude response):

$$|H(j\omega)G(j\omega)| = \left|\frac{1 - \alpha\, e^{-j\,\omega\, t_1}}{1 - \frac{1}{\alpha}\, e^{-j\,\omega\, t_1}}\right| = \frac{\left|1 - \alpha\, e^{-j\,\omega\, t_1}\right|}{\left|1 - \frac{1}{\alpha}\, e^{-j\,\omega\, t_1}\right|} = \frac{\left|1 - \alpha\cos(\omega\, t_1) + j\,\alpha\sin(\omega\, t_1)\right|}{\left|1 - \frac{1}{\alpha}\cos(\omega\, t_1) + j\,\frac{1}{\alpha}\sin(\omega\, t_1)\right|}$$

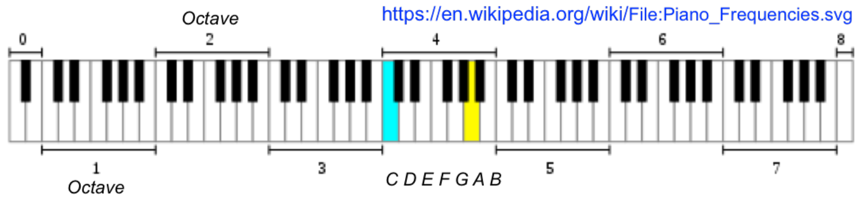
$$|H(j\omega)G(j\omega)| = \frac{\sqrt{(1 - \alpha\cos(\omega\, t_1))^2 + \alpha^2\sin^2(\omega\, t_1)}}{\sqrt{\left(1 - \frac{1}{\alpha}\cos(\omega\, t_1)\right)^2 + \left(\frac{1}{\alpha}\right)^2 \sin^2(\omega\, t_1)}} = \sqrt{\frac{1 - 2\,\alpha\cos(\omega\, t_1) + \alpha^2}{1 - 2\left(\frac{1}{\alpha}\right)\cos(\omega\, t_1) + \left(\frac{1}{\alpha^2}\right)}}$$

$$|H(j\omega)G(j\omega)| = \sqrt{\frac{\alpha^2 - 2\,\alpha\cos(\omega\, t_1) + 1}{\frac{1}{\alpha^2}(\alpha^2 - 2\,\alpha\cos(\omega\, t_1) + 1)}} = \alpha$$

See Handout I on All-Pass Filters.

Problem 8. Discrete-Time Audio Effects. *14 points.* **Related to Mini-Projects #1 and #2.**

The notes on the Western scale on an 88-key piano keyboard grouped into octaves follow:

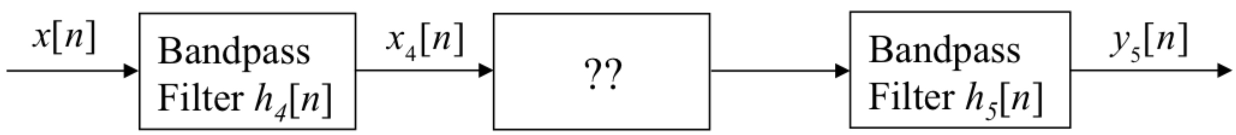


The frequency of note C6 (i.e. 'C' in the 6th octave) at 1046.5 Hz is twice the frequency of C5 at 523.25 Hz, and the frequency of C5 is twice the frequency of C4 at 261.625 Hz, and so forth.

This type of octave spacing occurs for all of the notes on the Western scale.

You are asked to design a **discrete-time** audio effects system that will extract each octave of frequencies and then alter that octave of frequencies to be in the next higher octave. All notes on the Western scale in the extracted octave should appear as the same notes in the next higher octave.

Here are the processing steps to extract the fourth octave in $x[n]$ and alter those frequencies to be in the fifth octave. Filter $h_k[n]$ represents the impulse response of the bandpass filter to extract the kth octave.



(a) For the bandpass filters, give an advantage of using finite impulse response (FIR) filters vs. infinite impulse response (IIR) filters. *3 points.*

HW #7 Prolog | *SPFirst p. 196*

<u>**Answer #1:**</u> **Linear phase FIR filters have constant group delay across all frequencies, whereas IIR filters have group delays that vary with frequency.**

<u>**Answer #2:**</u> **FIR filters are always BIBO stable. IIR filters are not always BIBO stable.**

<u>**Answer #3:**</u> **FIR filters would be somewhat easier to implement because they don't have to store or process previous output values at IIR filters would do.**

(b) For the bandpass filters, give an advantage of using IIR filters vs. FIR filters. *3 points*.

<u>**Answer #1:**</u> **For octave bandpass filters, IIR filters would have fewer coefficients and hence lower implementation complexity than FIR filters. The last page of the Midterm #2 solutions showed an IIR filter being 27 times more efficient to meet an octave bandpass specification.**

<u>**Answer #2:**</u> **Manual design can be easier for IIR filters because we can place both poles and zeros, whereas for FIR filters, we can only place zeros.**

<u>**Answer #3:**</u> **IIR filters have lower group delay than linear phase FIR filters.**

(c) What operation (or operations) would you choose for the ?? block. How does your choice guarantee that any note on the Western scale in the fourth octave in $x_4[n]$ will appear as the same note one octave higher in $y_5[n]$? What additional audio effects would your choice create? *8 points*.

The ?? block cannot be linear and time-invariant (LTI) because we are creating new frequencies on the output that weren't on the input. An LTI answer received 0 points.

<u>**Answer #1:**</u> **Any nonlinearity that would create a harmonic at twice each input frequency would make sure any note in the 4th octave would appear as the same note in the 5th octave:**

(a) A squaring block would double each input frequency and create a DC term. The doubling of frequency will make sure that each note appearing in the fourth octave would be converted into the same note in the fifth octave. Also, the fifth octave bandpass filter would filter out the DC term (at 0 rad/sample). Here's the effect of the squaring block on a single tone (frequency) at $\hat{\omega}_0$ rad/sample:

$$\cos^2(\hat{\omega}_0\, n) = \frac{1}{2} + \frac{1}{2}\cos(2\,\hat{\omega}_0\, n)$$

Additional audio effects would be due to cross-terms (called intermodulation products) of the principal frequencies in $x_4[n]$. For example,

$$\left(\cos(\hat{\omega}_0\, n) + \cos(\hat{\omega}_1\, n)\right)^2 = \cos^2(\hat{\omega}_0\, n) + 2\cos(\hat{\omega}_0\, n)\cos(\hat{\omega}_1\, n) + \cos^2(\hat{\omega}_1\, n)$$

will produce frequencies at $2\,\hat{\omega}_0$, $|\hat{\omega}_0 - \hat{\omega}_1|$, $\hat{\omega}_0 + \hat{\omega}_1$, $2\,\hat{\omega}_1$, and 0. The bandpass filter for the fifth octave will filter out 0 and $|\hat{\omega}_0 - \hat{\omega}_1|$.

(b) Frequency modulation such as $\cos(x_4[n])$ would produce many harmonics in the fifth octave for each note in the fourth octave. See Mini-Project #1 solutions.

Answer #2: Use sinusoidal amplitude modulation $x_4[n]\cos(\hat{\omega}_0\, n)$ where $\hat{\omega}_0 = 2\pi \frac{f_5 - f_4}{f_s}$ to shift the center frequency of the 4th octave (*f_4*) to that of the 5th octave (*f_5*). The notes in the fourth octave will not line up with notes in the fifth octave.

Answer #3: (Provided by two students) Remove every other sample and keep the sampling rate (playback rate) the same. This is called downsampling, and it does in fact have the intended effect of double frequency components. For example, let

$$x_4[n] = \cos(\hat{\omega}_0\, n)$$

where $\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}$. By removing every other sample, we'll produce a new signal $v[m]$ and we'll need a new discrete-time index *m* so that

$$v[m] = x_4[2m] = \cos\left(2\pi \frac{f_0}{f_s}(2m)\right) = \cos\left(2\pi \frac{2f_0}{f_s} m\right)$$

The frequency has doubled if the sampling rate *f_s* were to remain the same. The additional audio effect is that the signal when played back at the same sampling rate would only play for half of the amount of time in seconds as the original.